INTERNAL STRESSES IN AN ANISOTROPIC ELASTIC MEDIUM

(VNUTRENNIE NAPRIAZHENIIA V ANIZOTROPNOI UPRUGOI SREDE)

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The first section of this paper attempts to formulate a mathematical model of an elastic anisotropic medium with sources of internal stresses, such as dislocations, internodal atoms and vacancies, nonuniform temperature fields, etc. In this effort, we make essential use of certain ideas which seem to have been first enunciated by Kondo [1] and Kröner [2]. These references also cite other works in a similar direction.

The second section of the paper gives an explicit solution of a problem posed by Kröner on the invariant decomposition of a bi-valient tensor field into potential birotational components (deformation and incompatibility in the terminology of Kröner). At the same time we examine a more general decomposition of the stress tensor which may be interpreted as a decomposition into body-force stresses and internal stresses. Further, we determine and give an explicit expression for the Green's tensor of internal stresses for an unbounded anisotropic medium.

1. The geometry of an elastic continuum with sources of internal stresses. Qualitative characteristics of the continuum. We begin with the concept of "nearness". Physically, this means that two nearby particles in an initial state will remain nearby in any other arbitrary state. It is clear that this requirement is satisfied by the elastic deformations in a medium with a crystal lattice and that it is not satisfied by the displacements of sand. The concept of "nearness" is mathematically equivalent to the assumption that the medium, considered as a set of material points, is a topological space. Besides, two topological spaces are considered equivalent and indistinguishable if there exists a reciprocal single-valued and continuous transformation of one space onto the other. Such a transformation is called a homeomorphism (*). It may be said that a topological space is determined to within the accuracy of a homeomorphism.

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*) The exact definition of such concepts as topology, homeomorphism, manifold, etc. and their properties may be found, for rxample, in [3].

The description of the state of a material medium in a space is in practice impossible without the introduction of a system of coordinates. This, as well as other considerations require that a system of coordinates might be introduced at least in the neighborhood of each point. Nevertheless, in the general case there is no basis for requiring the existence of a single system of coordinates for the entire space, if spaces are considered which are topologically not equivalent to a Euclidean space, for example, a sphere, a torus, etc.

We shall assume that the neighborhood of every point of the medium is homeomorphic to an n-dimensional Euclidian space (or a half-space for a medium with a boundary). Usually $n = 2$ or $n = 3$, even though it will be sometimes convenient to examine the case $n > 3$.

Finally, in order to examine fields of sufficiently smooth functions in the medium (for example, differentiable or analytic functions), it will be necessary to require corresponding smoothness of the medium itself. Graphically this may be thought of in the following way: If two curves (or surfaces) in the medium have a specified order of tangency in the initial state, then they have the same order of tangency in an other arbitrary state (a medium such as clay of course does not have such a property). However this is possible only if the allowed transformations are not arbitrary homeomorphisms but are only sufficiently smooth diffeomorphisms. It turns out that without any essential restrictions on generality the latter may be considered analytic.

The requirements listed above may be succinctly formulated in a single postulate: the material medium is a differentiable (analytic) manifold.

The exterior space and the exterior metric. For definiteness we shall assume that the medium is homeomorphic to a three-dimensional Euclidean space E_3. We denote points in the medium by ξ and points in the space by x. Let Φ be a certain fixed smooth imbedding (diffeomorphism) of the medium in E_3

$$\Phi: \quad \xi \to x = \Phi(\xi) \tag{1.1}$$

By assumption there exists an inverse diffeomorphism

$$\Phi^{-1}: \quad x \to \xi = \Phi^{-1}(x) \tag{1.2}$$

We shall say that the given Φ specifies the external geometric state of the medium. All of the characteristics of the medium which depends only on Φ we shall call external (geometric) characteristics or functions of the external state.

For an actual given Φ we introduce a Lagrangian system of coordinates ξ^α, associated with the medium, and an Eulerian system of coordinates x^i. Then

$$\Phi: \quad x^i = x^i(\xi^\alpha), \qquad \Phi^{-1}: \quad \xi^\alpha = \xi^\alpha(x^i) \tag{1.3}$$

Here $x^i(\xi^\alpha)$ are sufficiently smooth functions with Jacobians which differ from zero.

The basic external characteristic of the medium, the external metric, we shall define as the distance between the points of E_3 in which are found the corresponding points of the medium in the state Φ

$$(ds^\Gamma)^2 = g^\Gamma_{\alpha\beta}(\xi)\, d\xi^\alpha d\xi^\beta = g^\Gamma_{ik}(x)\, dx^i dx^k \tag{1.4}$$

Here g^Γ_{ik} is the Euclidean metric tensor of E_3. If, in particular, we take Cartesian coordinates for x^i, then

$$(ds^\Gamma)^2 = \delta_{ik} dx^i dx^k = \sum_i (dx^i)^2 \tag{1.5}$$

Thus the external metric $(ds^\Gamma)^2$ uniquely induces an imbedding Φ and has various representations in Lagrangian and Eulerian coordinates.

We have the obvious relationships

$$g^\Gamma_{\alpha\beta}(\xi) = g^\Gamma_{ik}(x(\xi)) \frac{\partial x^i}{\partial \xi^\alpha} \frac{\partial x^k}{\partial \xi^\beta}, \qquad g^\Gamma_{ik}(x) = g^\Gamma_{\alpha\beta}(\xi(x)) \frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\beta}{\partial x^k} \tag{1.6}$$

It is easy to show that the converse assertion is also true. The given external metric uniquely determines Φ to within the accuracy of a rigid body motion. Naturally, in this case the external metric cannot be prescibed arbitrarily, but must satisfy Equation

$$R_{\lambda\mu\nu\rho}(g^\Gamma_{\alpha\beta}) = 0 \tag{1.7}$$

where $R_{\lambda\mu\nu\rho}$ is the Riemann-Christoffel curvature tensor. As is known, this is a nonlinear, second order differential operator acting on $g^\Gamma_{\alpha\beta}$ (see for example [4]).

The internal state and the internal metric. We shall call internal characteristics of the medium those characteristics which do not depend on the imbedding Φ. The collection of all internal characteristics determine the internal state of the medium. In particular, the above indicated qualitative characteristics relate to the number of internal characteristics: topology, nearness, smoothness. However, an inelastic medium may also possess all these properties. We shall attempt to describe those internal characteristics which differentiate an elastic medium from an inelastic one. In other words, we shall define the concept of elasticity.

Let ξ be a point in the medium and let U_ξ be its neighborhood. We imagine U_ξ to be cut out of the medium and isolated from all of the externally acting forces, but in this process we shall assume that the temperature of U_ξ is nonvarying. Then U_ξ will be in a certain external state Φ_0 which is specified to within the accuracy of the motion of U_ξ as a rigid body. We shall say that Φ_0 is the natural state of the neighborhood U_ξ. More exactly, under natural state we shall understand the limiting state when $U_\xi \to 0$. We shall assume that such a limit exists and does not depend upon the means of the approach of U_ξ to a point.

Let $\overset{\circ}{g}_{\alpha\beta}$ be the metric of the neighborhood U_ξ in the natural state. If we carry out a similar experiment for the neighborhood of every point, then we shall find $\overset{\circ}{g}_{\alpha\beta}(\xi)$ as a function of the point in the medium. We shall assume that this function is sufficiently smooth. In such a way we construct a metric

$$(ds^\circ)^2 = \overset{\circ}{g}_{\alpha\beta}(\xi)\, d\xi^\alpha d\xi^\beta = \overset{\circ}{g}_{ik}(x)\, dx^i dx^k \tag{1.8}$$

which by definition does not depend on Φ and, consequently, is an internal characteristic. We call this the internal metric of the medium (*).

The medium will be considered to be elastic only when under the conditions of the problem considered its internal state does not change.

We note that the natural state of the medium in the large, generally speaking, does not exist. This means that the internal metric, in contrast to the external metric, will not be Euclidean, and the corresponding curvature tensor will be different to zero in the general case. Hence the internal geomery of an elastic medium will be a Riemannian geometry.

E l a s t i c   s t r a i n . The elastic strain of a medium $\varepsilon_{\alpha\beta}$ is conveniently defined as the measure of derivation of the external state from the natural state. By definition, we set

$$\varepsilon_{\alpha\beta}(\xi) = \tfrac{1}{2}\left[g^{\Gamma}_{\alpha\beta}(\xi) - \overset{\circ}{g}_{\alpha\beta}(\xi)\right] \tag{1.9}$$

Figuratively speaking the elastic deformed state of the medium is the difference of the external and internal states. The strain $\varepsilon_{\alpha\beta}$ for the prescribed internal metric cannot be arbitrary but must satisfy Equation (1.7) after the substitution $g^{\Gamma}_{\alpha\beta} = \overset{\circ}{g}_{\alpha\beta} + 2\varepsilon_{\alpha\beta}$. This equation is a generalization of the well-known Saint-Venant compatibility conditions for the strains. The above clarifies the geometric meaning of the compatibility conditions.

If it is assumed that the natural state exists for the entire medium in the large, then in this state the internal metric coincides with the external and hence is Euclidean. Then, selecting a Lagrangian system of coordinates, in a corresponding manner, we may set

$$(ds^\circ)^2 = \delta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta \tag{1.10}$$

We specify the external state Φ by Equation

$$\Phi: \quad x^i = \delta_\alpha{}^i \xi^\alpha + u^i(\xi^\alpha) \tag{1.11}$$

Here u^i may be interpreted as the displacement vector for the transition from Φ_0 to Φ. Taking into account (1.6), we find for the external metric in the state Φ the expression

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*) The internal metric seems to have been introduced for the first time in the investigation by Kondo [1].

$$g^{\Gamma}_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha u^\gamma \partial_\beta u_\gamma \qquad \left(\partial_\alpha = \frac{\partial}{\partial \xi^\alpha}\right) \tag{1.12}$$

Substituting (1.12) into (1.9) we obtian the usual expression of ε in terms of u .

In the case of small deformations this takes on the form

$$\varepsilon_{\alpha\beta} = {}^1\!/_2\,(\partial_\beta u_\alpha + \partial_\alpha u_\beta) \qquad \text{or} \qquad \varepsilon = \operatorname{def} u \tag{1.13}$$

The equation of compatibility (1.7) also takes on the usual form in this case. Hence, the definition of the elastic strain ε by means of (1.9) is a natural generalization of the usual definition (1.13).

T h e o p e r a t o r Rot a n d t h e d e n s i t y o f s o u r c e s o f i n t e r n a l s t r e s s e s . In the sequel we shall not assume a Euclidean character of internal metric and shall examine a most important case when the metric may be assumed to deviate slightly from Euclidean metric. This means that there exists a system of coordinates ξ^α, in which the internal metric can be represented in the form

$$g_{\alpha\beta}^{\circ}(\xi) = \delta_{\alpha\beta} + 2\varepsilon_{\alpha\beta}^{\circ}(\xi), \quad |\varepsilon_{\alpha\beta}^{\circ}| \ll 1 \tag{1.14}$$

Here we also assume smaliness of the first two derivatives $\overset{\circ}{\varepsilon}_{\alpha\beta}$. In other terms, we assume a sufficient smallness of the strains and the displacements so that we may remain within the scope of a linear theory, in analogy to the classical linear theory of elasticity.

The first consequence of these simplifications will be the possibility of not making distinctions between Lagrangian and Eulerian systems of coordinates. The error which is caused by this process is of second order. It will also be convenient to assume a Cartesian coordinate system. The external and internal metrics, the deformation, the curvature tensor, etc. may be considered as corresponding tensor fields in E_3. Finally, in the expression for $R_{\lambda\mu\nu\rho}$ we may neglect nonlinear terms and consider the curvature tensor as a linear operator. If one takes into account its symmetry properties, then with it in E_3 may be reciprocally and uniquely associated a certain second order linear differential operator which acts on bi-valent tensors and whose values likewise are bi-valent tensors. We denote it by the symbol Rot and define by the relationship

$$p = \operatorname{Rot} q, \qquad p^{\alpha\beta} = \varepsilon^{\alpha\lambda\mu}\varepsilon^{\beta\nu\rho}\partial_\lambda\partial_\nu q_{\mu\rho} \tag{1.15}$$

Here $\varepsilon^{\alpha\beta\gamma}$, as usual, denotes the antisymmetrical unit pseudo-tensor.

The operator Rot may also be written in the form

$$\operatorname{Rot} = \operatorname{rot}\,(\operatorname{rot})' \tag{1.16}$$

where rot is the usual curl operator and the prime denotes transposition.

This operator was first introduced into the theory of elasticity by Krutkov [5]. The clarification of its geometrical meaning and the application to the continuum theory of dislocations is basically due to Kröner [2] (the latter denoted the operator by the symbol Ink).

We note some important properties of this operator. From its definition it follows that Rot commutes with the transposition operator and hence with the symmetrization and alternation operators. Likewise the following relations are obvious

$$\operatorname{div}\operatorname{Rot} = 0, \qquad \operatorname{Rot}\operatorname{def} = 0 \tag{1.17}$$

As will be obvious from the following, the operator Rot has in a certain sense the same values for a field of tensors of second valency as does the operator rot for vector fields. The notation that has been used for this operator in the present paper makes this clear. It seems appropriate to call this the double rotor (birotor) and to call the tensor field p, represented in the form $p = \operatorname{Rot} q$, birotational. The corresponding tensor field of the form $\operatorname{grad} u$ or $\operatorname{def} u$, where u is a vector, could be called a potential field.

Equation (1.7) may now be rewritten in the form

$$\operatorname{Rot} g^{\Gamma} = 0 \tag{1.18}$$

In the case of internal Euclidean metric, Equation

$$\operatorname{Rot} \varepsilon = 0 \tag{1.19}$$

will be a compact form of writing the usual Saint-Venant compatibility conditions. From the above it follows that these are the necessary and sufficient conditions for an elastic deformation ε (at least locally) to be representable in the form $\operatorname{def} u$.

In the case of a noneuclidean internal metric we follow Kröner and introduce the notation

$$\operatorname{Rot} g^{\circ} = -2\eta \tag{1.20}$$

Therefore

$$\operatorname{Rot} \varepsilon = \eta \tag{1.21}$$

Here $\eta_{\alpha\beta}$ is the density of sources of internal stresses. Kröner calls the tensor $\eta_{\alpha\beta}$ the incompatibility tensor. Obviously the condition $\eta = 0$ is the necessary and sufficient condition for the absence of internal stresses.

We note two important cases. If the internal stresses are caused by a nonuniform distribution of temperatures, hen from the above it follows immediately that

$$\eta = \operatorname{Rot} T, \qquad T_{\alpha\beta} = \gamma\theta(x)\,\delta_{\alpha\beta} \tag{1.22}$$

Here $\theta(x)$ is the temperature and γ is the coefficient of thermal expansion. If the internal stresses are stipulated by a distribution of dislocations, then, as Kröner [2] showed,

$$\eta = S\,(\operatorname{rot}\alpha)' \tag{1.23}$$

where α is Burgers' mass flux density vector and S is the symmetrization operator.

2. Green's tensor of internal stresses. Since we are limiting our investigation to small deformations, it is natural to assume that Hooke's law is valid

$$\sigma^{\alpha\beta} = C^{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}, \qquad \varepsilon_{\lambda\mu} = \overset{*}{C}{}^{-1}_{\lambda\mu\alpha\beta}\,\sigma^{\alpha\beta}, \qquad C^{-1\alpha\beta\lambda\mu}\overset{*}{C}_{\lambda\mu\nu\rho} = \delta_{\nu}^{(\alpha}\delta_{\rho}^{\beta)} \tag{2.1}$$

We shall rewrite these relations in straightforward notation

$$\sigma = C\varepsilon, \qquad \varepsilon = C^{-1}\sigma \tag{2.2}$$

The equations which determine stresses in an anisotropic medium without sources of internal stress have the form

$$\operatorname{div}\sigma = -f, \qquad \operatorname{Rot} C^{-1}\sigma = 0 \tag{2.3}$$

where f is the density of body forces. As follows from the preceding, analogous equations in the case of the presence only of sources of internal stresses are written in the form

$$\operatorname{div}\sigma = 0, \qquad \operatorname{Rot} C^{-1}\sigma = \eta \qquad (\operatorname{div}\eta = 0) \tag{2.4}$$

The general case is obtained by superposition.

We shall examine solutions (2.3) and (2.4) vanishing at infinity, for an infinite anisotropic medium. We shall assume that f and η are locally integrable and that they are either different from zero in a bounded region or decrease sufficiently rapidly at infinity. It is known that the solution (2.3) may be represented in the form

$$\sigma^{\alpha\beta}(x) = \int G_i^{\alpha\beta}(x - x_0) f^i(x_0)\, dx_0 \tag{2.5}$$

where $G_i^{\alpha\beta}(x)$ is the Green's tensor of the theory of elasticity for stresses. When there is no possibility of ambiguity, it is convenient to use the symbol $*$ for the convolution integral of two functions over the entire space. Then (2.5) can be rewritten as

$$\sigma = G * f \tag{2.6}$$

where it is necessary to take into account that in addition to the convolution integration we also have here a tensor contraction over one index.

Obviously, the Green's tensor G must satisfy Equations

$$\partial_\alpha G_i^{\alpha\beta} = -\delta_i^\beta\,\delta(x), \qquad \operatorname{Rot} C^{-1}_{\lambda\mu\alpha\beta}\, G_i^{\alpha\beta} = 0 \tag{2.7}$$

or in straightforward notion

$$\operatorname{div} G = -e, \qquad \operatorname{Rot} C^{-1}G = 0 \tag{2.8}$$

where $e(x)$ is the kernel of the identity operator acting over the vectors.

The second equation is satisfied identically if we set

$$G_i^{\alpha\beta}(x) = C^{\alpha\beta\lambda\mu}\partial_\lambda U_{i\mu}(x) \tag{2.9}$$

Here $U_{i\mu}(x)$ is the Green's tensor for displacements. Substituting (2.9) into the first equation, we obtain an equation which determines $U_{i\mu}$

$$C^{\alpha\beta\lambda\mu}\partial_\beta\partial_\lambda U_{i\mu} = -\delta_i^\alpha\,\delta(x) \tag{2.10}$$

As was shown by Lifshits and Rozentsveig [6], in the general case the construction of $U^i_{\ \mu}$ reduces to the determination of the roots of an algebraic equation of sixth degree whose coefficients are determined by the elastic constant tensor $C^{\alpha\beta\lambda\mu}$. In a number of cases, for example, for an isotropic medium, for hexagonal symmetry, and for all sorts of weak anisotropy, the solution of this equation may be written in explicit form. In the remaining cases it is necessary to solve this equation numerically. In what follows we shall assume that the Green's tensors $U_{i\mu}$ and $G_i^{\alpha\beta}$ are known. We mention in passing that $U_{i\mu}$ decreases at infinity like r^{-1} and, therefore, $G_i^{\alpha\beta}$ like r^{-2}

The goal of this section is the construction of the Green's tensor for the system of equations (2.4). However, the concept of Green's tensor in the present case has not as yet been defined since the right-hand side of (2.4) must satisfy the additional condition $\operatorname{div} \eta = 0$. Hence, as a preliminary, we examine a subsidiary problem which is also of intrinsic interest.

The decomposition of a tensor field into invariant components. As Kröner [2] has shown, a sufficiently smooth symmetrical bi-valent tensor A, vanishing at infinity, may be uniquely decomposed into a potential component A_1° and a birotational component A_2°

$$A = A_1^\circ + A_2^\circ; \quad \begin{matrix} \operatorname{Rot} A_1^\circ = 0, & A_1^\circ = \operatorname{def} b \\ \operatorname{div} A_2^\circ = 0, & A_2^\circ = \operatorname{Rot} B \end{matrix} \tag{2.11}$$

Here b and B are respectively a vector and a symmetrical bi-valent tensor playing the role of potentials. To define B uniquely, one may in this case superpose the additional condition $\operatorname{div} B = 0$. However Kröner did not give an algorithm with which one could find the potentials b and B and effect the indicated decomposition in practice.

It is convenient to formulate this problem in the following manner. We introduce on the space of the tensors A the projection operators Π° and Θ°, determined from the relationships

$$\begin{gathered} \Pi^\circ\Pi^\circ = \Pi^\circ, \qquad \Theta^\circ\Theta^\circ = \Theta^\circ, \qquad \Pi^\circ + \Theta^\circ = E \\ \operatorname{Rot} \Pi^\circ = 0, \qquad \operatorname{div} \Theta^\circ = 0 \end{gathered} \tag{2.12}$$

where E is the identity operator. When the entire space decomposes into a direct sum of two subspaces of potential and birotational tensors

$$A_1^\circ = \Pi^\circ A = \pi^\circ * A, \qquad A_2^\circ = \Theta^\circ A = \vartheta^\circ * A \tag{2.13}$$

Here $\pi^\circ(x)$ and $\vartheta^\circ(x)$ are fourth-valent tensors which are the kernels of the corresponding projection operators. They are to be taken in the sense of generalized functions [7].

Thus, the problem consists of finding explicit expressions for Π° and Θ° or, what is the same, for π° and ϑ°.

We introduce the generalized Kroneker tensor

$$\varepsilon_{\beta\mu}^{\alpha\lambda} = \delta_\beta^\alpha \delta_\mu^\lambda - \delta_\mu^\alpha \delta_\beta^\lambda \tag{2.14}$$

and rewrite the algebraic identity

$$\varepsilon_{\beta_1\mu_1}^{\alpha_1\lambda_1} \varepsilon_{\beta_2\mu_2}^{\alpha_2\lambda_2} = \delta_{\beta_1}^{\alpha_1}\delta_{\beta_2}^{\alpha_2}\delta_{\mu_1}^{\lambda_1}\delta_{\mu_2}^{\lambda_2} + \delta_{\mu_1}^{\alpha_1}\delta_{\mu_2}^{\alpha_2}\delta_{\beta_1}^{\lambda_1}\delta_{\beta_2}^{\lambda_2} - \delta_{\beta_1}^{\alpha_1}\delta_{\mu_2}^{\alpha_2}\delta_{\mu_1}^{\lambda_1}\delta_{\beta_2}^{\lambda_2} - \delta_{\mu_1}^{\alpha_1}\delta_{\beta_2}^{\alpha_2}\delta_{\beta_1}^{\lambda_1}\delta_{\mu_2}^{\lambda_2} \tag{2.15}$$

Contracting it with $\partial_{\lambda_1\lambda_2}\partial^{\mu_1\mu_2}$, we obtain the operator identity

$$\varepsilon^{\alpha_1\lambda_1}_{\beta_1\mu_1}\varepsilon^{\alpha_2\lambda_2}_{\beta_2\mu_2}\partial_{\lambda_1\lambda_2}\partial^{\mu_1\mu_2} = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\triangle^2 + \partial_{\beta_1\beta_2}\partial^{\alpha_1\alpha_2} - \triangle\,(\delta^{\alpha_1}_{\beta_1}\partial_{\beta_2}\partial^{\alpha_2} + \delta^{\alpha_2}_{\beta_2}\partial_{\beta_1}\partial^{\alpha_1}) \quad (2.16)$$

As is not difficult to verify, the following relationship is valid

$$\varepsilon^{\alpha\lambda}_{\beta\mu} = \varepsilon_{\beta\mu\nu}\varepsilon^{\nu\alpha\lambda} \quad (2.17)$$

Remembering the definition (1.15) of the Rot operator and taking into account (2.17), after some transformation, the identity (2.16) can be finally written in the form

$$\triangle^2 = \operatorname{def}\,(2\triangle - \operatorname{grad}\operatorname{div})\operatorname{div} + \operatorname{Rot}\operatorname{Rot} \quad (2.18)$$

This identity is analogous to the well-known formula of vector analysis

$$\triangle = \operatorname{grad}\operatorname{div} - \operatorname{rot}\operatorname{rot} \quad (2.19)$$

We set

$$R^{\alpha_1\alpha_2}_{\beta_1\beta_2} = r\,(x)\,\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} \quad (r^2 = \delta_{\lambda\mu}x^\lambda x^\mu) \quad (2.20)$$

and apply both sides of the identity (2.18) to the expression

$$-\frac{1}{8\pi}R^{\alpha_1\alpha_2}_{\beta_1\beta_2} * A_{\alpha_1\alpha_2}$$

Taking into account that

$$\triangle^2 r\,(x) = -\,8\pi\delta(x) \quad (2.21)$$

we obtain

$$A = -\frac{1}{8\pi}\operatorname{def}\,(2\triangle - \operatorname{grad}\operatorname{div})\operatorname{div} R * A - \frac{1}{8\pi}\operatorname{Rot}\operatorname{Rot} R * A \quad (2.22)$$

It is easy to see that the generalized tensor functions

$$\pi^\circ = -\frac{1}{8\pi}\operatorname{def}\,(2\triangle - \operatorname{grad}\operatorname{div})\operatorname{div} R, \qquad \vartheta^\circ = -\frac{1}{8\pi}\operatorname{Rot}\operatorname{Rot} R \quad (2.23)$$

$$(\pi^{\circ\alpha_1\alpha_2}_{\beta_1\beta_2} + \vartheta^{\circ\alpha_1\alpha_2}_{\beta_1\beta_2} = \delta^{(\alpha_1}_{\beta_1}\delta^{\alpha_2)}_{\beta_2}\delta\,(x))$$

are the kernels of the unknown projection operator, which solves the posed problem on the decomposition of a tensor field into potential and birotational components. Simultaneously, expressions have been obtained for the vector and tensor potentials

$$b = \frac{r}{8\pi} * (\operatorname{grad}\operatorname{div} - 2\triangle)\operatorname{div} A, \qquad B = -\frac{r}{8\pi} * \operatorname{Rot} A$$
$$\operatorname{div} B = 0 \quad (2.24)$$

We note that the preceeding in an analogous way it is easy to generalize the given decomposition to the case of asymmetrical bi-valent tensors.

T h e G r e e n ' s t e n s o r f o r i n t e r n a l s t r e s s e s . We represent the solution of the system (2.4) in the form

$$\sigma^{\alpha\beta} = H^{\alpha\beta}_{\lambda\mu} * \eta^{\lambda\mu} \quad (2.25)$$

It is clear that the Green's tensor H must satisfy the first equation of the system. As far as the second equation for H is concerned, on its right-hand side should stand the kernel of an operator which coincides with the identity operator on a subspace of birotational tensors and which has zero divergence. But it is immediately apparent that these properties are possessed by Θ°. Hence,

$$\operatorname{div} H = 0, \qquad \operatorname{Rot} C^{-1}H = \theta^\circ \tag{2.26}$$

Above, we constructed an algorithm for the decomposition of a tensor into potential and birotational components. We consider a decomposition of a more general form. Let A, as before, be a symmetric bi-valent tensor vanishing at infinity. We set

$$A = A_1 + A_2, \qquad \operatorname{Rot} C^{-1}A_1 = 0, \qquad \operatorname{div} A_2 = 0 \tag{2.27}$$

or in terms of the projection operators

$$A_1 = \Pi A, \quad A_2 = \Theta A; \quad \Pi\Pi = \Pi, \quad \Theta\Theta = \Theta, \quad \Pi + \Theta = E$$
$$\operatorname{Rot} C^{-1}\Pi = 0, \qquad \operatorname{div} \Theta = 0 \tag{2.28}$$

This decomposition may be interpreted in the following way. Let A be the stress tensor. Then A_1 is the component of stresses which is caused by body forces, that is, a solution of the system (2.3), whereas A_2 are internal stresses, that is a solution of the system (2.4). The decomposition (2.12) may in this case be considered as a particular instance of the given decomposition if one sets

$$C^{\alpha\beta}_{\cdot\cdot\lambda\mu} = \delta_\lambda^{(\alpha}\delta_\mu^{\beta)} \tag{2.29}$$

It is easy to verify that one of the possible representations Π and Θ is given by Expressions

$$\Pi = -G * \operatorname{div}, \qquad \Theta = E + G * \operatorname{div} \tag{2.30}$$

where G is the Green's tensor defined above from the theory of elasticity. However, this representation still cannot be directly used for the construction of H.

Of fundamental importance is the operator identity

$$\Theta = \Theta C \Theta^\circ C^{-1} \tag{2.31}$$

For the proof of this identity we first of all mention that the divergence of both sides of the equation, by virtue of (2.28), vanishes. On the other hand, from (2.12) and (2.28) it follows that

$$\operatorname{Rot} \Theta^\circ = \operatorname{Rot}, \qquad \operatorname{Rot} C^{-1}\Theta = \operatorname{Rot} C^{-1} \tag{2.32}$$

Now applying the operator $\operatorname{Rot} C^{-1}$ to both sides of Equation (2.31) and taking into account (2.32), we obtain an identity. However if the result of the application of the operators div and $\operatorname{Rot} C^{-1}$ to two tensors, vanishing at infinity, coincide, then the tensors themselves coincide, since their difference is a solution of the homogeneous equations of the theory of elasticity vanishing at infinity. This however proves the identity (2.31).

Thus, for the internal stresses σ we have

$$\sigma = \Theta\sigma = \Theta C \Theta^{\circ} C^{-1}\sigma = -\frac{1}{8\pi}(C + G * \operatorname{div} C)(\operatorname{Rot} \operatorname{Rot} R * C^{-1}\sigma) \qquad (2.33)$$

From the properties of convolution [7] it follows

$$\operatorname{Rot} \operatorname{Rot} R * C^{-1}\sigma = \operatorname{Rot} \operatorname{Rot} (r * C^{-1}\sigma) = \\ = \operatorname{Rot} (r * \operatorname{Rot} C^{-1}\sigma) = \operatorname{Rot} R * \operatorname{Rot} C^{-1}\sigma \qquad (2.34)$$

Substituting into (2.33) we find

$$\sigma = -\frac{1}{8\pi}(C \operatorname{Rot} R + G * \operatorname{div} C \operatorname{Rot} R) * \operatorname{Rot} C^{-1}\sigma \qquad (2.35)$$

Therefore for the Green's tensor of the internal stresses we obtain Expression

$$H(x) = -\frac{1}{8\pi}[C \operatorname{Rot} R(x) + G(x) * \operatorname{div} C \operatorname{Rot} R(x)] \qquad (2.36)$$

which solves the problem that has been posed. By a direct verification one may be convinced that H satisfies Equations (2.26). It is likewise easy to see that $H \sim r^{-1}$ for $x \to \infty$.

The problem of determining the internal stresses in an isotropic medium was solved earlier by another method by Kröner [2].

In conclusion we remark that the general problem of finding the stress tensor in an infinite anisotropic medium in the presence of external and internal sources of stress may be formulated in the following way: It is required to find the symmetric tensor σ under the conditions $\sigma(\infty) = 0$, if we have prescribed

$$\operatorname{div} \sigma = -f, \qquad \operatorname{Rot} C^{-1}\sigma = \eta \qquad (2.37)$$

The solution can be written in the form

$$\sigma = G * f + H * \eta \qquad (2.38)$$

In the particular case where C satisfies (2.29), it follows from (2.23)

$$G^{\circ} = \frac{1}{8\pi} \operatorname{def} (\operatorname{grad} \operatorname{div} - 2\Delta) R, \qquad H^{\circ} = -\frac{1}{8\pi} \operatorname{Rot} R \qquad (2.39)$$

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